

Gradient: direction of maximal increase

Critical point: a point P in the domain of f where either $\nabla f(P) = 0$ or $\nabla f(P)$ does not exist

Fermat's Extrema Theorem: If f has a local extreme value at P , then P is a critical point of f

Extreme Value Theorem: If f is defined on a closed and bounded subset $U \subseteq \mathbb{R}^n$, then f obtains its global extrema on U .

closed and bounded: A set is closed and bounded iff it is a union of finitely many closed and bounded intervals in \mathbb{R}^1



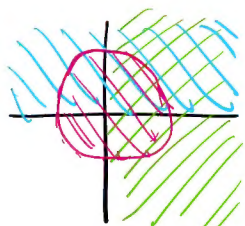
This suggests a method for optimizing global values on a closed and bounded subset

Alg (compact set method) let f be a function defined on a closed and bounded subset U . To compute global extrema of f on U ,

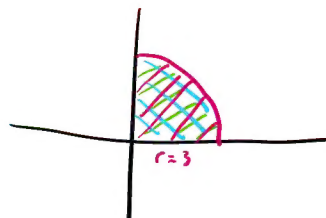
- 1) compute critical points of f on U
- 2) compute the max and min of f on the critical points
- 3) optimize f along the boundary of U

The max/min is global extrema values of f on U

Example: find the global extrema of $f(x,y) = x^2y$ on $U = \{(x,y), 0 \leq x, 0 \leq y, x^2+y^2 \leq 3\}$



$$0 \leq y$$
$$0 \leq x$$
$$x^2 + y^2 \leq 3$$



Step 1: compute critical points in U

$$\nabla f = (y^2, 2xy) \quad \text{so } \nabla f = 0 \text{ iff } (y^2, 2xy) = (0, 0)$$
$$y^2 = 0 \quad \text{iff } y = 0$$

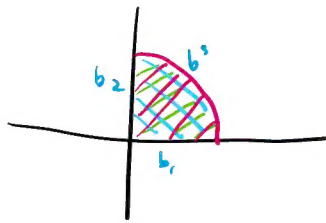
$$2xy = 0 \quad \text{iff } x = 0 \text{ or } y = 0$$

$$\text{So, true iff } \nabla f = 0$$

Note: in this example the boundary points all intersect with U therefore, we will skip

Step 3: Analyze the boundaries

Step 3: Analyze the constraints



Since $r=3$, b_1 or b_2 will be $\sqrt{3}$

Parametrize

$$b_1(t) = (t, 0) \text{ on } 0 \leq t \leq \sqrt{3}$$

$$b_2(t) = (0, t) \text{ on } 0 \leq t \leq \sqrt{3}$$

$$b_3(t) = (\sqrt{3} \cos(t), \sqrt{3} \sin(t)) \text{ on } 0 \leq t \leq \frac{\pi}{2}$$

u. Optimization

$$\text{for } b_1 \quad f(b_1(t)) = f(t, 0) = t \cdot 0^2 = 0$$

$$\text{for } b_2 \quad f(b_2(t)) = f(0, t) = 0 \cdot t^2 = 0$$

$$\text{for } b_3 \quad f(b_3(t)) = f(\sqrt{3} \cos(t), \sqrt{3} \sin(t)) = (\sqrt{3} \cos(t))(\sqrt{3} \sin(t))^2 = 3\sqrt{3} \cos(t) \sin^2(t)$$

$$\text{So, } g'(t) = 3\sqrt{3} - \sin(t) \sin^2(t) + \cos(t) \cdot 2 \sin(t) \cos(t) = 3\sqrt{3} \sin(t) (2 \cos^2(t) - \sin^2(t))$$

$$g'(t) = 0 \text{ if } \sin(t) = 0 \text{ or } 2 \cos^2(t) - \sin^2(t) = 0$$

$$2 \cos^2(t) = \sin^2(t) \quad \text{divide by } \cos^2(t) \text{ since } \cos(t) \neq 0$$

$$\tan^2(t) = 2$$

$$\tan(t) = \pm \sqrt{2}$$

$$\text{if } t = 0 \quad \text{or } t = \arctan(\sqrt{2}) \text{ or } t = \arctan(-\sqrt{2})$$

less than $\frac{\pi}{2}$,
reject

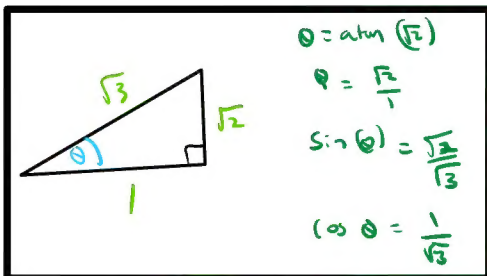
$$t = 0 \quad \text{or } t = \arctan \sqrt{2}$$

\therefore Test $g(t)$

$$g(0) = 3\sqrt{3} \cos(0) \sin(0) = 0$$

$$g\left(\frac{\pi}{2}\right) = 3\sqrt{3} \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) = 0$$

$$g(\arctan \sqrt{2}) = 3\sqrt{3} \cos(\arctan \sqrt{2}) \sin(\arctan \sqrt{2}) = 2$$



$$3\sqrt{3} \left(\frac{1}{\sqrt{3}}\right) \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = 2$$

The abs max of $u = 2$

The abs min of $u = 0$

First Derivative Test

let f be diff at critical point \vec{p}

first derivative test

let f be diff at critical point \vec{p}

① if $D_{\vec{u}}f(\vec{p} + \epsilon \vec{u}) > 0$ for all \vec{u} should $\epsilon > 0$ and all unit vectors \vec{u} , then f has a local min at \vec{p} .

② if $D_{\vec{u}}f(\vec{p} + \epsilon \vec{u}) < 0$ for all \vec{u} should $\epsilon > 0$ and all unit vectors \vec{u} , then f has a local max at \vec{p} .

- This is too hard to use on a problem

Second derivative test

Warning! failure is possible

$$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx} = f_{xx} \cdot f_{yy} - f_{xy}^2$$

Definition

This only works as stated for $f(x,y)$

1) If $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) = f_{yy}(\vec{p}) \cdot f_{xx}(\vec{p}) - f_{xy}(\vec{p})^2 > 0$ then \vec{p} is a point with local min of f

2) If $f_{xx}(\vec{p}) < 0$ and $D(\vec{p}) = f_{yy}(\vec{p}) \cdot f_{xx}(\vec{p}) - f_{xy}(\vec{p})^2 > 0$ then \vec{p} is a point with local max of f

3) If $D(\vec{p}) = f_{yy}(\vec{p}) \cdot f_{xx}(\vec{p}) - f_{xy}(\vec{p})^2 = 0$ then \vec{p} is a saddle point